Towards a classification of Lorentzian holonomy groups.
Part II: Semisimple, non-simple weak-Berger algebras

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Abstract

The holonomy group of an $(n+2)$–dimensional simply-connected, indecomposable but non-irreducible Lorentzian manifold (M, h) is contained in the parabolic group $(\mathbb{R} \times SO(n)) \ltimes \mathbb{R}^n$. The main ingredient of such a holonomy group is the $SO(n)$–projection $G := pr_{SO(n)}(Hol_p(M, h))$ and one may ask whether it has to be a Riemannian holonomy group. In this paper we show that this is always the case, completing our results of [Lei03]. We draw consequences for the existence of parallel spinors on Lorentzian manifolds.

Contents

1 Introduction

This paper is an addendum to our paper [Lei03] where we gave a partial classification of reduced holonomy groups of indecomposable Lorentzian manifolds. The holonomy group of an $(n+2)$-dimensional, indecomposable, non-irreducible Lorentzian manifold is contained in the parabolic group whose Lie algebra is $(\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$. Concerning the three projections, L. Berard-Bergery and A. Ikemakhen distinguished in [BI93] four different types of indecomposable subalgebras of $(\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$. But the main ingredient of such a holonomy algebra is the $\mathfrak{so}(n)$-projection. Also in [BI93] a Borel-Lichnerowicz-type decomposition property is proved:

1.1 Theorem. *[BI93] Let be $\mathfrak{g} = pr_{\mathfrak{so}(n)}\mathfrak{hol}(M^{n+2}, h)$. Then it holds: For the decomposition of the representation space $\mathbb{R}^n = E_0 \oplus \ldots \oplus E_k$ of $\mathfrak{g}$ into irreducible components there is a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k \subset \mathfrak{so}(n)$ into ideals and each of these $\mathfrak{g}_i$ acts irreducibly on E_i and trivial on E_j for $i \neq j$.*

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Thus for the problem of classifying possible $\mathfrak{so}(n)$-parts of indecomposable Lorentzian holonomy algebras, one can restrict oneself to the study of irreducibly acting subalgebras of $\mathfrak{so}(n)$. For this we introduced in [Lei02a] the notion of a weak-Berger algebra and showed that the $\mathfrak{so}(n)$-component of an indecomposable Lorentzian holonomy algebra is a weak-Berger algebra and all its irreducibly acting components too. Using this we classified the unitary acting weak-Berger algebras. In the first part [Lei03] to the present paper we extended the classification to irreducible weak-Berger algebras which are simple, obtaining the following theorem:

1.2 Theorem. *[Lei02a, Theorem 3.2], [Lei03, Theorem 3.21 and Theorem 4.7] Let $\mathfrak{g} \subset \mathfrak{so}(n, \mathbb{R})$ be a real, irreducible weak-Berger algebra which is simple or acts unitary. Then it is a Riemannian holonomy algebra.*

For this we used the distinction of real representations into representations of real and of non-real type. Orthogonal representation of non-real type are unitary representations. For representations of real type which are weak-Berger the complexified Lie algebra (with the complexified representation of course) is also weak-Berger. Furthermore holds that an orthogonal Lie algebra of real type has to be semisimple. Hence [Lei03] leaves open the following problem: Classify all complex, irreducible weak-Berger algebras which are semisimple but not simple. We will solve this problem in the present paper. It uses very much results and proofs of [Lei03]. Here we will complete the proof of the following theorem by completing the classification result in the semisimple case.

1.3 Theorem. *Any weak-Berger algebra is the holonomy algebra of a Riemannian holonomy algebra. In particular: The $\mathfrak{so}(n)$-component of an indecomposable, non-irreducible Lorentzian holonomy algebra is a Riemannian holonomy algebra.*

To complete the proof in the semisimple case we will show the following:

1.4 Proposition. *Any irreducibly acting, semisimple, non-simple complex weak-Berger algebra is the compexification of Riemannian holonomy algebra.*

To prove this statement we will use several results of [Lei03] which describe the weak-Berger property in terms of root systems.
Up to dimension $n = 9$ the result of theorem 1.3 was proved by A. Galaev in [Gal03], partially using results of [Lei02a].
At the end of this paper we will show some applications for Lorentzian manifolds with parallel spinors showing that their holonomy group is of the form $G \ltimes \mathbb{R}^n$ where G is a holonomy group of a Riemannian manifold with parallel spinor.

2 Proof of the result

2.1 The notion of a weak-Berger algebra

First we recall the notion of a weak-Berger algebra.

2.1 Definition. Let $\mathfrak{g} \subset \mathfrak{so}(E,h)$ be an orthogonal Lie algebra. Then $\mathfrak{g}$ is is called **weak-Berger algebra** if $\mathfrak{g} = span\{Q(x)|x \in E, Q \in \mathcal{B}_h(\mathfrak{g})\}$, where $\mathcal{B}_h(\mathfrak{g}) := \{Q \in E^* \otimes \mathfrak{g}|h(Q(x)y,z) + h(Q(y)z,x) + h(Q(z)x,y) = 0\}$.

If E is a real vector space and the the real Lie algebra $\mathfrak{g}_0 \subset \mathfrak{so}(E,h)$ is irreducible of real type, i.e. the complexification is irreducible, then $\mathfrak{g}_0$ is a weak-Berger algebra if and only if $\mathfrak{g} := \mathfrak{g}_0^{\mathbb{C}} \subset \mathfrak{so}(E^{\mathbb{C}}, h^{\mathbb{C}})$ is a complex weak-Berger algebra. Since an irreducibly acting, complex $\mathfrak{g} \subset \mathfrak{so}(V,H))$ — for V a complex vector space — is semisimple we can use the tools of root space and weight space decomposition to classify complex weak-Berger algebras.

We denote by Ω the weights of $\mathfrak{g} \subset \mathfrak{so}(V,H)$, by Δ the roots of $\mathfrak{g}$ and we set $\Delta_0 = \Delta \cup \{0\}$. If $\alpha \in \Delta$ the we define $\Omega_\alpha := \{\mu \in \Omega | \mu + \alpha \in \Omega\}$. Then it holds:

2.2 Proposition. *[Lei03, Proposition 2.6] Let $\mathfrak{g}$ be a semisimple Lie algebra with roots Δ and $\Delta_0 = \Delta \cup \{0\}$. Let $\mathfrak{g} \subset \mathfrak{so}(V,H)$ irreducible, weak-Berger with weights Ω. Then the following properties are satisfied:*

(PI) *There is a $\mu \in \Omega$ and a hyperplane $U \subset \mathfrak{t}^*$ such that*

$$\Omega \subset \{\mu + \beta \mid \beta \in \Delta_0\} \cup U \cup \{-\mu + \beta \mid \beta \in \Delta_0\}. \tag{1}$$

(PII) *For every $\alpha \in \Delta$ there is a $\mu_\alpha \in \Omega$ such that*

$$\Omega_\alpha \subset \{\mu_\alpha - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{-\mu_\alpha + \beta \mid \beta \in \Delta_0\}. \tag{2}$$

Furthermore we obtained:

2.3 Proposition. *[Lei03, Proposition 2.13] Let $\mathfrak{g} \subset \mathfrak{so}(V,H)$ be an irreducible complex weak-Berger algebra. Then there is an extremal weight Λ such that one of the following properties is satisfied:*

(SI) *There is a pair (Λ, U) with an extremal weight Λ a hyperplane U in $\mathfrak{t}^*$ such that every extremal weight different from Λ and $-\Lambda$ is contained in U and $\Omega \subset \{\Lambda + \beta \mid \beta \in \Delta_0\} \cup U \cup \{\Lambda + \beta \mid \beta \in \Delta_0\}$.*

(SII) *There is an $\alpha \in \Delta$ such that $\Omega_\alpha \subset \{\Lambda - \alpha + \beta \mid \beta \in \Delta_0\} \cup \{-\Lambda + \beta \mid \beta \in \Delta_0\}$.*

There is a fundamental system such that the extremal weight in (SI) and (SII) is the highest weight.

In the following we classified all simple Lie algebras which satisfy (PI) and (PII) or (SI) or (SII) and showed that they are Riemannian holonomy algebras. In the present paper we will do this for semisimple Lie algebras which are not simple.

2.2 Semisimple, non-simple weak-Berger algebras

From now on let $\mathfrak{g}$ be a complex semisimple, non-simple Lie algebra, irreducible represented on a complex vector space V. To a decomposition of $\mathfrak{g}$ into ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ corresponds a decomposition of the irreducible module V into factors $V = V_1 \otimes V_2$ which are irreducible $\mathfrak{g}_1$- resp. $\mathfrak{g}_2$-modules. $X = (X_1, X_2) \in \mathfrak{g}$ acts as follows: $X \cdot (v_1 \otimes v_2) = (X_1 \cdot v_1) \otimes v_2 + v_1 \otimes X_2 \otimes v_2$. The Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is the sum of the Cartan subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. If Δ are the roots of $\mathfrak{g}$ and Δ^i the roots of $\mathfrak{g}_i$ then $\Delta = \Delta^1 \cup \Delta^2$. For the weights it holds $\Omega = \Omega^1 + \Omega^2$. Analogously we denote for $\alpha \in \Delta^i$ the set Ω_α^i. Then holds the following

2.4 Lemma. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a semisimple Lie algebra, $V = V_1 \otimes V_2$ be an irreducible representation of it. If $\alpha \in \Delta^1$ then it holds*

$$\Omega_\alpha = \Omega_\alpha^1 + \Omega^2 \tag{3}$$

Proof. For $\lambda \in \Omega_\alpha$ we have $\Omega \ni \lambda + \alpha = \lambda_1 + \alpha + \lambda_2$ with $\lambda_i \in \Omega_i$. Hence $\lambda_1 + \alpha \in \Omega^1$. If otherwise $\lambda_1 + \alpha \in \Omega^1$ then $\lambda_1 + \lambda_2 + \alpha \in \Omega$, i.e. $\lambda_1 + \lambda_2 \in \Omega_\alpha$. $\square$

Assuming the weak-Berger property we get by this:

2.5 Lemma. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a semisimple Lie algebra, $V = V_1 \otimes V_2$ be an irreducible representation of it which is weak-Berger. If the dimensions of V_1 and V_2 are greater than 2, then for any $\alpha \in \Delta^i$ the set Ω_α^i contains at most 2 elements.*

Proof. Suppose that $dim\ V_2 \geq 3$, i.e. $\#\Omega^2 \geq 3$. Let $\alpha \in \Delta^1$, $\lambda_1 \in \Omega_\alpha^1$ and $\lambda_2 \in \Omega^2$, i.e. $\lambda_1 + \lambda_2 \in \Omega_\alpha^1$. Now from the property (PII) follows that there is a $\mu_\alpha =: \mu_\alpha^1 + \mu_\alpha^2 \in \Omega$ such that $\lambda_1 + \lambda_2 = \mu_\alpha - \alpha + \beta$ or $\lambda_1 + \lambda_2 = -\mu_\alpha + \beta$ with $\beta \in \Delta_0 = \Delta^1 \cup \Delta^2 \cup \{0\}$. If now $\#\Omega^2 \geq 3$ and $\#\Omega_\alpha^1 \geq 3$ then we can choose $\lambda_1 \neq \mu_\alpha^1 - \alpha$, $\lambda_1 \neq -\mu_\alpha^1$ and $\lambda_2 \neq \pm\mu_\alpha^2$. This gives a contradiction. $\square$

Now we can use a result of L. Schwachhöfer from [Sch99].

2.6 Proposition. *[Sch99, Lemma 3.23] Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ be an irreducibly acting, semisimple subalgebra. If for any α the set Ω_α contains at most two elements, then $\mathfrak{g}$ is conjugate to one of the following representations:*

1. *$\mathfrak{sl}(n, \mathbb{C})$ acting on $\mathbb{C}^n$; in this case Ω_α is a singleton for all $\alpha \in \Delta$.*

2. *$\mathfrak{so}(n, \mathbb{C})$ acting on $\mathbb{C}^n$; in this case Ω_α contains two elements for all $\alpha \in \Delta$, and their sum equals to $-\alpha$.*

3. *$\mathfrak{sp}(n, \mathbb{C})$ acting on $\mathbb{C}^{2n}$; in this case Ω_α contains two elements $\alpha \in \Delta$ is short, and their sum equals to $-\alpha$, and $\Omega_\alpha = \{-\frac{1}{2}\alpha\}$.*

From this result we obtain the following corollary, proving proposition 1.4 if the dimensions of the factors of V are greater than 2.

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2.7 Corollary. *Let* $\mathfrak{g} \subset \mathfrak{so}(V,h)$ *be a complex, semisimple, non-simple, irreducibly acting weak-Berger algebra. If* $\mathfrak{g}$ *decomposes into* $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *such that for the corresponding decomposition of* $V = V_1 \otimes V_2$ *holds that* $\dim V_i \geq 3$ *for* $i = 1, 2$, *then it holds:* $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{so}(m, \mathbb{C})$ *acting on* $\mathbb{C}^n \otimes \mathbb{C}^m$, *or* $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) \oplus \mathfrak{sp}(m, \mathbb{C})$ *acting on* $\mathbb{C}^{2n} \otimes \mathbb{C}^{2m}$. *In particular it is the complexification of a Riemannian holonomy representation of a symmetric space of type BDI resp. CII (for the types see [Hel78].*

Proof. By lemma 2.5 it must hold $\#\Omega_\alpha^i \leq 2$ for both summands. So we have to built sums of the Lie algebras of proposition 2.6. But only the sum of two orthogonal acting Lie algebras, or a sum of two symplectic acting Lie algebras acts orthogonal. $\quad\square$

By this result we are left with semisimple Lie algebras where the irreducible representation of one summand is two-dimensional, i.e. $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_2$ and $V = \mathbb{C}^2 \otimes V_2$. Since we are interested in $\mathfrak{g} \subset \mathfrak{so}(V, h)$ and $\mathfrak{sl}(2, \mathbb{C})$ acts symplectic on $\mathbb{C}^2$ the representation of $\mathfrak{g}_2$ on V_2 has to be symplectic too.
In this situation we prove the following

2.8 Proposition. *Let* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_2$ *be a semismple, complex weak-Berger algebra, acting irreducibly on* $\mathbb{C}^2 \otimes V_2$. *Then* $\mathfrak{g}_2 \subset \mathfrak{sp}(V_2)$ *satisfies the following properties:*

(PIII) *There is a* $\mu \in \Omega^2$ *and an affine hyperplane* $A \subset \mathfrak{t}_2^*$ *such that*

$$\Omega^2 \subset \left\{ \mu + \beta \mid \beta \in \Delta_0^2 \right\} \cup A \cup \{-\mu\}. \tag{4}$$

(PIV) *There is a* $\mu \in \Omega^2$ *such that*

$$\Omega^2 \subset \left\{ \mu + \beta \mid \beta \in \Delta_0^2 \right\} \cup \left\{ -\mu + \beta \mid \beta \in \Delta_0^2 \right\}. \tag{5}$$

Proof. Since $\mathfrak{g}$ is weak-Berger it satisfies the properties (PI) and (PII). We draw the consequences from both for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{g}_2$. For the representation of $\mathfrak{sl}(2, \mathbb{C})$ on $\mathbb{C}^2$ we have that $\Omega^1 = \{\Lambda, -\Lambda\}$. Let $\alpha \in \Delta^1 = \{\alpha, -\alpha\}$ be the positive root of $\mathfrak{sl}(2, \mathbb{C})$. Hence $\Omega_\alpha^1 = \{-\Lambda\}$, since $-\Lambda = \Lambda - \alpha$.

(PIII) $\mathfrak{g}$ satisfies the property (PI) with a hyperplane $U := (T_1 + T_2)^\perp$ and a weight $\mu = \mu_1 + \mu_2$. Let $\lambda = \mu_1 + \lambda_2 \in \Omega$ be a weight of $\mathfrak{g}$. If λ lies in a hyperplane of $\mathfrak{t} = \mathfrak{t}_1 \oplus^\perp \mathfrak{t}_2$, then $0 = \langle \mu_1, T_1 \rangle + \langle \lambda_2, T_2 \rangle$, i.e. λ_2 lies in an affine hyperplane of $\mathfrak{t}$. If $\lambda = \mu + \beta = \mu_1 + \mu_2 + \beta$, then $\lambda_2 = \mu_2 + \beta$ with $\beta \in \Delta^2$. If $\lambda = -\mu + \beta = -\mu_1 - \mu_2 + \beta$, then β has to be in Δ^1 and $\lambda_2 = \mu_2$. Hence $\mathfrak{g}_2 \subset \mathfrak{sp}(V_2)$ satisfies (PIII).

(PIV) $\mathfrak{g}$ satisfies the property (PII). Suppose that α is the positive root of $\mathfrak{sl}(2, \mathbb{C})$. Then $\Omega_\alpha^1 = \{-\Lambda\}$ and $\Omega_\alpha = \{-\Lambda\} \cup \Omega^2$. Now let $\lambda \in \Omega^2$, i.e. $-\Lambda + \lambda \in \Omega_\alpha$. By (PII) there is a $\mu_\alpha = \mu^1 + \mu^2$ such that $-\Lambda + \lambda = \mu_\alpha - \alpha + \beta$ or $-\Lambda + \lambda = -\mu_\alpha + \beta$ with $\beta \in \Delta_0 = \Delta^1 \cup \Delta^2 \cup \{0\}$. Since $\mu_1 = \pm\Lambda$ this implies $\lambda \in \{\mu_2 + \beta | \beta \in \Delta_0^2\} \cup \{-\mu_2 + \beta | \beta \in \Delta_0^2\}$, i.e. (PIV) is satisfied.

So we have shown that both, (PIII) and (PIV) are satisfied. $\square$

2.9 Example. We set $\mathfrak{g}_2 = \mathfrak{sl}(2,\mathbb{C})$ and check if $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ acting on $\mathbb{C}^2 \otimes V_2$ is a weak-Berger algebra. This is to check whether $\mathfrak{sl}(2,\mathbb{C})$ acting on V_2 satisfies (PIII) and (PIV) and is symplectic. To be symplectic means that the representations has an even number of weights, (PIV) implies that V_2 has at most 6 weights but (PIII) implies that V_2 has at most 4 weights. Hence the only weak-Berger algebras with the structure of $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ are those acting on $\mathbb{C}^4$ and on $\mathbb{C}^2 \otimes sym_0^3 \mathbb{C}^2 = \mathbb{C}^8$. Both are of course complexifications of Riemannian holonomy representations, the first of the 4-dimensional symmetric space of type CII, i.e. $Sp(2)/Sp(1) \cdot SP(1)$ and the second of the 8-dimensional symmetric space of type GI, i.e. $G_2/SU(2) \cdot SU(2)$ (in the compact case, see [Hel78]).

Now we try to reduce the problem in a way that we only have to deal with simple Lie algebras.

2.10 Lemma. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a semisimple, complex Lie algebra acting irreducibly on V, satisfying the property (PIV). Then $\mathfrak{g}$ is simple or $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{g}_2$ acting on $\mathbb{C}^2 \otimes V_2$.*

Proof. Suppose that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and that $\#\Omega^1 \geq 3$. Let $\mu = \mu_1 + \mu_2$ be the weight from the property (PIV). We consider a weight $\lambda = \lambda_1 + \lambda_2 \in \Omega = \Omega^1 + \Omega^2$ with $\lambda_1 \neq \pm\mu_1$. Then (PIV) implies that $\lambda_2 = \mu_2$ or $\lambda_2 = -\mu_2$, i.e. $\#\Omega^2 \leq 2$. This implies the proposition of the lemma. $\square$

To complete the reduction we need a further

2.11 Lemma. *Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{g}_3$ be a semisimple complex Lie algebra, acting irreducibly on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes V_3$ and satisfying the property (PII). Then for any root $\alpha \in \Delta^3$ of $\mathfrak{g}_3$ holds $\#\Omega_\alpha^3 \leq 2$.*

Proof. Let $\alpha \in \Delta^3$ and $\mu_\alpha^1 + \mu_\alpha^2 + \mu_\alpha^3$ the weight from the property (PII). Then $\Omega_\alpha = \Omega^1 + \Omega^2 + \Omega_\alpha^3 \ni -\mu_\alpha^1 + \mu_\alpha^2 + \lambda$ with $\lambda \in \Omega_\alpha^3$ arbitrary. Again (PII) implies $\lambda = \mu_\alpha^3 - \alpha$ or $\lambda = -\mu_\alpha^3$, i.e. $\#\Omega_\alpha^3 \leq 2$. $\square$

Both lemmata give the following result.

2.12 Proposition. *Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) + \mathfrak{g}_2$ be a semisimple, complex Lie algebra, acting irreducibly on $\mathbb{C}^2 \otimes V_2$ which is supposed to be weak-Berger. Then $\mathfrak{g}_2$ is simple, acts irreducible and symplectic on V_2 satisfying (PIII) and (PIV), or $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C}) = \mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$ acting irreducibly on $\mathbb{C}^4 \otimes \mathbb{C}^n$.*

Proof. The proof is obvious by lemma 2.10 and lemma 2.11 and the result of proposition 2.6 keeping in mind that $\mathfrak{g}$ is orthogonal, hence $\mathfrak{g}_2$ is symplectic and $\mathfrak{g}_3$ has to be orthogonal again. $\square$

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Of course, the representation of $\mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(n,\mathbb{C})$ on $\mathbb{C}^4 \otimes \mathbb{C}^n$ is the complexification of a Riemannian holonomy representation of the symmetric space of type BDI.

2.3 Simple Lie algebras satisfying (PIII) and (PIV)

In this section we deal with the remaining problem to classify complex, simple irreducibly acting symplectic Lie algebras with the property (PIII) and (PIV).

2.13 Proposition. *Let $\mathfrak{g} \subset \mathfrak{sp}(V)$ be simple, irreducibly acting and satisfying (PIV). Then it satisfies (SII).*

Proof. First we note that the fact that the representation is symplectic leaves us with the simple Lie algebras with root systems A_n, B_n, C_n, D_n and E_7. In particular the Lie algebra of type G_2 is excluded. This implies that for two roots α and β it holds that $\left| \frac{\langle \alpha, \beta \rangle}{\|\alpha\|^2} \right| \in \{1, \frac{1}{2}, 0\}$, a fact which we will use several times in the following proof.
Let μ be the weight from the property (PIV). We consider two cases.

Case 1: μ is not an extremal weight: In this case there is a root $\alpha \in \Delta$ such that $\mu + \alpha = \Lambda$ is extremal. We show that (SII) is satisfied with the tripel $(\Lambda, -\Lambda, \alpha)$.

We suppose that (SII) is not satisfied, i.e. there is a $\lambda \in \Omega_\alpha \subset \Omega$ such that neither $\lambda = \Lambda - \alpha + \beta$ nor $\lambda = -\Lambda + \beta$ for a $\beta \in \Delta$. $\lambda \in \Omega$ and $\lambda + \alpha \in \Omega$ gives by (PIV) that $\lambda = -\Lambda + \alpha + \beta$ with $\beta \in \Delta$ and $\alpha + \beta \notin \Delta_0$, as well as $\lambda = \Lambda - 2\alpha + \gamma$ with $\gamma \in \Delta$ and $\alpha - \gamma \notin \Delta_0$. By properties of root systems this implies that $\langle \alpha, \beta \rangle \geq 0$ and $\langle \alpha, \gamma \rangle \leq 0$. Furthermore it is

$$2\Lambda = 3\alpha + \beta - \gamma. \tag{6}$$

Now it is $\frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = 3 + \frac{\langle \beta, \alpha \rangle}{\|\alpha\|^2} - \frac{\langle \gamma, \alpha \rangle}{\|\alpha\|^2} \geq 3$, entailing $\Lambda - 3\alpha \in \Omega$. Since $\Lambda - 3\alpha \neq \Lambda - \alpha + \delta$ for a $\delta \in \Delta_0$. (PIV) implies $\Lambda - 3\alpha \neq -\Lambda + \alpha + \delta$, i.e.

$$2\Lambda = 4\alpha + \delta, \tag{7}$$

with $\delta \neq -\alpha$. (6) and (7) give

$$0 = \alpha + \delta + \gamma - \beta. \tag{8}$$

Now suppose that $\frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = 3$, i.e. $\langle \beta, \alpha \rangle = \langle \gamma, \alpha \rangle = 0$. In this case (7) gives $\frac{2\langle \delta, \alpha \rangle}{\|\delta\|^2} = -2$ and therefore $\frac{2\langle \Lambda, \delta \rangle}{\|\delta\|^2} = -3$. This implies that $\Lambda + 3\delta \in \Omega$, but this is together with (7) is a contradiction to (PIV).

Now suppose that $\frac{2\langle \Lambda, \alpha \rangle}{\|\alpha\|^2} = 4$, i.e. $\frac{\langle \beta, \alpha \rangle - \langle \gamma, \alpha \rangle}{\|\alpha\|^2} = 1$. Then (8) implies $\langle \alpha, \delta \rangle = 0$. $\Lambda - 4\alpha \in \Omega$ implies by (PIV) and $3\alpha \notin \Delta$ that $2\Lambda = 5\alpha + \varepsilon$, i.e. $\alpha - \delta \in \delta$. Since $\langle \alpha, \delta \rangle = 0$ this implies that α and δ are short roots and $\alpha - \delta$ is a long one, i.e. $\frac{\|\delta\|^2}{\|\alpha - \delta\|^2} = \frac{\|\alpha\|^2}{\|\alpha - \delta\|^2} = \frac{1}{2}$. But this gives that $\frac{2\langle \Lambda, \alpha - \delta \rangle}{\|\alpha - \delta\|^2} = \frac{5}{2}$ which is a contradiction.

Finally suppose that $\frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2} \geq 5$. Hence $\frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2} - \frac{\langle\gamma,\alpha\rangle}{\|\alpha\|^2} \geq 2$ On the other hand $\Lambda - 5\alpha \in \Omega$ and by (PIV) $2\alpha - \delta \in \Delta$. This implies that $\frac{2\langle\alpha,\delta\rangle}{\|\alpha\|^2} \geq 2$. But both inequalities are a contradiction to (8).

Case 2. $\mu := \Lambda$ *is an extremal weight.* To proceed analogously as in the first case we fix a root $\alpha \in \Delta$, which is supposed to be long in case of root systems with roots of different lenght, and we show that (SII) is satisfied for the tripel $(\Lambda, -\Lambda, \alpha)$.

Again we suppose that (SII) is not satisfied, i.e. there is a $\lambda \in \Omega_\alpha \subset \Omega$ such that neither $\lambda = \Lambda - \alpha + \beta$ nor $\lambda = -\Lambda + \beta$ for a $\beta \in \Delta$. $\lambda \in \Omega$ and $\lambda + \alpha \in \Omega$ gives by (PIV) that $\lambda = \Lambda + \beta$ with $\beta \in \Delta$ and $\alpha + \beta \notin \Delta_0$, as well as $\lambda = -\Lambda - \alpha + \gamma$ with $\gamma \in \Delta$ and $\alpha - \gamma \notin \Delta_0$. By properties of root systems this implies that $\langle\alpha,\beta\rangle \geq 0$ and $\langle\alpha,\gamma\rangle \leq 0$. Since α is supposed to be a long root this the same as $\frac{\langle\alpha,\beta\rangle}{\|\alpha\|^2} \in \{0, \frac{1}{2}\}$ and $\frac{\langle\alpha,\gamma\rangle}{\|\alpha\|^2} \in \{-\frac{1}{2}, 0\}$. Furthermore it is

$$2\Lambda = -\alpha - \beta + \gamma \tag{9}$$

and hence $\mathbb{Z} \ni \frac{2\langle\Lambda,\alpha\rangle}{\|\alpha\|^2} = 1 - \frac{\langle\beta,\alpha\rangle}{\|\alpha\|^2} + \frac{\langle\gamma,\alpha\rangle}{\|\alpha\|^2} =: a \leq -1$. Then of course $a \in \{-2, -1\}$.

First suppose that $a = -1$. In the case it is $\langle\alpha,\beta\rangle = \langle\alpha,\gamma\rangle = 0$. Then because of $\mathbb{Z} \ni \frac{2\langle\Lambda,\beta\rangle}{\|\beta\|^2} = -1 + \frac{\langle\beta,\gamma\rangle}{\|\beta\|^2}$ and $\mathbb{Z} \ni \frac{2\langle\Lambda,\gamma\rangle}{\|\gamma\|^2} = -1 + \frac{\langle\beta,\gamma\rangle}{\|\gamma\|^2}$ it must hold that $\frac{\langle\beta,\gamma\rangle}{\|\beta\|^2}$ and $\frac{\langle\beta,\gamma\rangle}{\|\gamma\|^2}$ are integers. But this can only be true if β and γ are both, long and short. This is impossible.

Now suppose that $a = -2$, i.e. $\frac{\langle\alpha,\beta\rangle}{\|\alpha\|^2} = \frac{1}{2}$ and $\frac{\langle\alpha,\gamma\rangle}{\|\alpha\|^2} = -\frac{1}{2}$. Then $\Lambda - 2\alpha \in \Omega$, i.e. by (PIV) we get that

$$2\Lambda = -2\alpha + \delta. \tag{10}$$

with $\delta \in \Delta_0$ with $\delta \neq \pm\alpha$ because otherwise we would get $a = -1$ or $a = -3$.

Now the existence of a root ε with the property $\|\delta\| \leq \|\varepsilon\|$ would give a contradiction since

$$\mathbb{Z} \ni \frac{2\langle\Lambda,\varepsilon\rangle}{\|\varepsilon\|^2} = -\underbrace{\frac{2\langle\alpha,\varepsilon\rangle}{\|\varepsilon\|^2}}_{\in\mathbb{Z}} + \underbrace{\frac{\langle\delta,\varepsilon\rangle}{\|\varepsilon\|^2}}_{\notin\mathbb{Z}}.$$

This implies that δ is a long root in the root system of type C_n. In C_n the system of long roots equals to $A_1 \times \ldots \times A_1$. By this $\frac{\langle\alpha,\beta\rangle}{\|\alpha\|^2} = \frac{1}{2}$ and $\frac{\langle\alpha,\gamma\rangle}{\|\alpha\|^2} = -\frac{1}{2}$ implies that β and γ are short roots recalling that α was supposed to be a long one.

But then by (9) we get

$$\frac{2\langle\Lambda,\beta\rangle}{\|\beta\|^2} = -\frac{\langle\alpha,\beta\rangle}{\|\beta\|^2} + \frac{\langle\beta,\gamma\rangle}{\|\beta\|^2} - 1 = -\frac{1}{2}\frac{\|\alpha\|^2}{\|\beta\|^2} + \frac{\langle\beta,\gamma\rangle}{\|\beta\|^2} - 1 = -2 + \frac{\langle\beta,\gamma\rangle}{\|\beta\|^2} \notin \mathbb{Z}$$

since β and γ are short. But this is a contradiction.

$\square$

As a consequence of this proposition we only have to check the irreducible representations of simple Lie algebra whether they satisfy (SII) — done in [Lei03] — and then to add the condition that the representations are symplectic — instead of orthogonal. We obtain the following result.

2.14 Proposition. *Let $\mathfrak{g} \subset \mathfrak{sp}(V)$ be a complex, simple, irreducibly and symplectic acting Lie algebra satisfying (PIII) and (PIV) and different from $\mathfrak{sl}(2, \mathbb{C})$. Then the root system and the highest weight of the representation are one of the following:*

1. *A_5: ω_3, i.e. $\mathfrak{g} = \mathfrak{sl}(6, \mathbb{C})$ acting on $\wedge^3 \mathbb{C}^6$.*

2. *C_n: ω_1, i.e. $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ acting on $\mathbb{C}^{2n}$.*

3. *C_3: ω_3, i.e. $\mathfrak{g} = \mathfrak{sp}(3, \mathbb{C})$ acting on $\mathbb{C}^{14}$.*

4. *D_6: ω_6, i.e. $\mathfrak{g} = \mathfrak{so}(12, \mathbb{C})$ acting on $\mathbb{C}^{32}$ as spinor representation.*

5. *E_7: ω_1, i.e. the standard representation of E_7 of dimension 56.*

Proof. (PIV) implies (SII), so we use former results checking the Lie algebras satisfying (SII) whether they are symplectic. For this we consider two cases.

First we suppose that $0 \in \Omega$. In proposition 3.6 and corollary 3.9 of [Lei03] it is proved that any such representation which satisfies (SII) and is self-dual is orthogonal. Hence if 0 is a weight, no symplectic representation satisfies (SII).

Now suppose that $0 \notin \Omega$. In the proof of proposition 3.18 of [Lei03] we have shown that the representations of the following Lie algebras with $0 \notin \Omega$ satisfy (SII). Now we check if these are symplectic and in some cases if they satisfy (PIII) and (PIV).

1. A_n with $n \leq 7$ odd, $\Lambda = \omega_{\frac{n+1}{2}}$. The only representation of these which is symplectic is the one for $n = 5$.

2. B_n: ω_n for $n \leq 7$ the spin representations, and $\omega_1 + \omega_2$ for $n = 2$. The latter is symplectic and the the former is symplectic for $n = 5, 6$. ($B_2 \simeq C_2$ we will study in the next point.) Now we show that these remaining representations does not satisfy (PIII) or (PIV).

   Of course the representation of B_2 with highest weight $\Lambda = \omega_1 + \omega_2 = \frac{3}{2}e_1 + \frac{1}{2}e_2$ can not satisfy (PIV) because it has 12 weights while B_2 has only 8 roots.

   The spin representation for $n = 6$ can not obey (PIV): W.l.o.g we may assume that Λ from (PIV) is the highest weight $\Lambda = \frac{1}{2}(e_1 + \ldots + e_6)$. But then for the weight $\lambda = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 - e_6)$ it holds neither $\Lambda - \lambda \in \Delta_0$ nor $\Lambda + \lambda \in \Delta_0$.

   The spin representation for $n = 5$ does satisfy (PIV) but not (PIII) since all the weights $\frac{1}{2}(\pm e_1 \pm \ldots \pm e_5)$ with 3 minus signs can not lie on the same affine hyper plane.

   Hence none of the symplectic representations satisfying (SII) satisfies (PIII) and (PIV).

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3. C_n with $\Lambda = \omega_1 + \omega_i$ or $\Lambda = \omega_i$. These are symplectic for i even in the first case and for i odd in the second case.

   Again we have to impose the condition (PIV) on both. First we consider the representation with highest weight $\Lambda = \omega_i = e_1 + \ldots + e_i$. Since the set of roots of C_n equals to $\{e_i \pm e_j, \pm 2e_k\}$ we get

   $$\Omega = \{\pm e_{k_1} \pm \ldots \pm e_{k_i}\} \cup \{\pm e_{k_1} \pm \ldots \pm e_{k_{i-2}}\} \cup \ldots \cup \{\pm e_k\}.$$

   From this one sees that (PIV) can not be satisfied if $n \geq 5$.

   With analogous considerations we exclude the case where $\Lambda = \omega_1 + \omega_i$ with i even.

4. D_n with $\Lambda = \omega_n$ and $n \leq 8$. But these are only symplectic for $n = 6$ and $n = 2$. The latter is excluded since $D_2 = A_1 \times A_1$, a case which is handled in the previous subsection.

5. For E_7 remains only the representation given in the proposition.

$\square$

If we now combine the result of this and the previous subsection we get the following:

2.15 Corollary. *Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{g}_2$ be a semisimple, complex weak-Berger algebra acting on $\mathbb{C}^2 \otimes V_2$. Then it is the complexification of a Riemannian holonomy representation, in particular the complexification of the holonomy representation of a non-symmetric $Sp(1) \cdot Sp(n)$-manifold or of the following Riemannian symmetric spaces (we list only the compact symmetric space):*

1. *Type EII: $E_6/SU(2) \cdot SU(6)$,*

2. *Type CII: $Sp(n+1)/Sp(1) \cdot Sp(n)$,*

3. *Type FI: $F_4/SU(2) \cdot Sp(3)$,*

4. *Type EVI: $E_7/SU(2) \cdot Spin(12)$,*

5. *Type EIX: $E_8/SU(2) \cdot E_7$*

and of type GI, i.e. $G_2/SU(2) \cdot SU(2)$.

This corollary together with corollary 2.7 proves proposition 1.4 and therefore theorem 1.3.

3 Consequences

In order to explain the conclusion more in detail we cite the result of L. Berard-Bergery and A. Ikemakhen about four different types of indecomposable, non-irreducible Lorentzian holonomy algebras. One considers $\mathbb{R}^n$ with the Minkowskian scalar product of the form $\eta := \begin{pmatrix} 0 & 0^t & 1 \\ 0 & E_n & 0 \\ 1 & 0^t & 0 \end{pmatrix}$. Any indecomposably, but non-irreducibly acting subalgebra of $\mathfrak{so}(\eta)$ is contained in the parabolic algebra $(\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$. Furthermore one can prove the following result.

3.1 Theorem. *[BI93] Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{so}(\eta)$ which acts indecomposably and non-irreducibly on $\mathbb{R}^{n+2}$, $\mathfrak{g} := pr_{\mathfrak{so}(n)}(\mathfrak{h})$ with $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{d}$ its Levi-decomposition in the center and the derived Lie algebra. Then $\mathfrak{h}$ belongs to one of the following types.*

1. *If $\mathfrak{h}$ contains $\mathbb{R}^n$, then we have the types*

    Type 1: *$\mathfrak{h}$ contains $\mathbb{R}$. Then $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$.*

    Type 2: *$pr_{\mathbb{R}}(\mathfrak{h}) = 0$ i.e. $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n$.*

    Type 3: *Neither Type 1 nor Type 2.*
    In that case there exists a surjective homomorphism $\varphi : \mathfrak{z} \to \mathbb{R}$, such that

    $$\mathfrak{h} = (\mathfrak{l} \oplus \mathfrak{d}) \ltimes \mathbb{R}^n$$

    where $\mathfrak{l} := graph\ \varphi = \{(\varphi(T), T) | T \in \mathfrak{z}\} \subset \mathbb{R} \oplus \mathfrak{z}$. Or written as matrices:

    $$\mathfrak{h} = \left\{ \left. \begin{pmatrix} \varphi(A) & v^t & 0 \\ 0 & A+B & -v \\ 0 & 0 & -\varphi(A) \end{pmatrix} \right| A \in \mathfrak{z}, B \in \mathfrak{d}, v \in \mathbb{R}^n \right\}.$$

2. *In case $\mathfrak{h}$ does not contain $\mathbb{R}^n$ we have **Type 4:***

    There exists

    (a) a non-trivial decomposition $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^l$, $0 < k, l < n$,

    (b) a surjective homomorphism $\varphi : \mathfrak{z} \to \mathbb{R}^l$

    such that $\mathfrak{g} \subset \mathfrak{so}(k)$ and $\mathfrak{h} = (\mathfrak{d} \oplus \mathfrak{l}) \ltimes \mathbb{R}^k \subset \mathfrak{p}$ where $\mathfrak{l} := \{(\varphi(T), T) | T \in \mathfrak{z}\} = graph\ \varphi \subset \mathbb{R}^l \oplus \mathfrak{z}$. Or written as matrices:

    $$\mathfrak{h} = \left\{ \left. \begin{pmatrix} 0 & \varphi(A)^t & v^t & 0 \\ 0 & 0 & A+B & -v \\ 0 & 0 & 0 & -\varphi(A) \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| A \in \mathfrak{z}, B \in \mathfrak{d}, v \in \mathbb{R}^k \right\}.$$

From this and theorem 1.3 we get an obvious corollary.

3.2 Corollary. *Let $\mathfrak{h}$ be the holonomy algebra of an indecomposable, non irreducible $n+2$-dimensional Lorentzian manifold.*

1. *If $\mathfrak{h}$ is of type 1 or 2, then it holds $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n$ or $\mathfrak{g} \ltimes \mathbb{R}^n$, where $\mathfrak{g}$ is a Riemannian holonomy algebra.*

2. *If $\mathfrak{h}$ is of type 3 or 4, then $\mathfrak{g} = pr_{\mathfrak{so}(n)}\mathfrak{h}$ is a Riemannian holonomy algebra with at least one irreducible factor equal to a Riemannian holonomy algebra with center, i.e. equal to $\mathfrak{so}(2)$ acting on $\mathbb{R}^2$ or on itself, $\mathfrak{so}(2) \oplus \mathfrak{so}(n)$ acting on $\mathbb{R}^{2n}$ $\mathfrak{so}(2) \oplus \mathfrak{so}(10)$ acting on $\mathbb{R}^{32}$ as the reellification of the complex spinor module of dimension 16, $\mathfrak{so}(2) \oplus \mathfrak{e}_6$ acting on $\mathbb{R}^{54}$, $\mathfrak{u}(n)$ acting on $\mathbb{R}^{2n}$ or on $\mathbb{R}^{n(n-1)}$.*

Regarding the epimorphisms $\varphi : \mathfrak{z}(\mathfrak{g}) \mapsto \mathbb{R}^k$ from theorem 3.1 there is a theorem of C. Boubel [Bou00, Théorème 3.IV.3 and Corollaire 3.IV.3] which describes how to construct a metric with holonomy of type 3 or 4 from metrics with holonomy of type 1 or 2, under certain algebraic conditions on $\mathfrak{g}$ of course. Knowing the possible $\mathfrak{g}$'s gives some candidates to start with in order to construct such metrics of type 3 and 4.

Finally we want to draw some conclusions about the existence of parallel spinor fields on Lorentzian manifold. The existence of a parallel spinor field on a Lorentzian spin manifold implies the existence of a parallel vector field which has to be lightlike or timelike. In the latter case the manifold splits by the de-Rham decomposition theorem (at least locally) into a factor $(\mathbb{R}, -dt^2)$ and Riemannian factors which are flat or irreducible with a parallel spinor, i.e. with holonomy $\{1\}$, G_2, $Spin(7)$, $Sp(k)$ or $SU(k)$.

In the case where the parallel vector field is lightlike we have a Lorentzian factor which is indecomposable, but with parallel lightlike vector field (and parallel spinor) and flat or irreducible Riemannian manifolds with parallel spinors. Hence in this case one has to know which indecomposable Lorentzian manifolds admit a parallel spinor. The existence of the lightlike parallel vector field forces the holonomy of such a manifold with parallel spinor to be contained in $\mathfrak{so}(n) \ltimes \mathbb{R}^n$ i.e. to be of type 2 or 4. Furthermore the spin representation of $\mathfrak{so}(n)$-projection $\mathfrak{g} \subset \mathfrak{so}(n)$ must admit a trivial subrepresentation (see for example [Lei02b]). Up to dimension $n+2 = 11$ these groups where described by R. L. Bryant in [Bry00] and J.M. Figueroa O'Farrill in [FO00] obtaining that at least the maximal ones are of type 2 and of the shape $(Riemannian\ holonomy) \ltimes \mathbb{R}^n$. Now we get this result in general.

3.3 Corollary. *Let $\mathfrak{h}$ be the holonomy algebra of an indecomposable Lorentzian spin manifold with parallel spinor field. Then $\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n$ where $\mathfrak{g}$ is the holonomy algebra of a Riemannian manifold with parallel spinor, i.e. a sum of the following algebras: $\{0\}$, $\mathfrak{g}_2$, $\mathfrak{spin}(7)$, $\mathfrak{sp}(k)$ or $\mathfrak{su}(k)$.*

Proof. We have to exclude that the holonomy algebra can be of type 4 under the assumption of a parallel spinor field. But if $\mathfrak{h}$ is of type 4, the $\mathfrak{so}(n)$-projection $\mathfrak{g}$ has

a $\mathfrak{so}(2)$-summand. (Also in the $\mathfrak{u}(n)$ case since $\mathfrak{u}(n) = \mathfrak{so}(2) \oplus \mathfrak{su}(n)$.) But a direct calculation ($\mathfrak{so}(2) = \mathbb{R}J$ with $J^2 = -id$) shows that the spin representation of such a $\mathfrak{so}(2)$-summand is an isomorphism of the spinor module, i.e. there can be no trivial subrepresentation. $\qquad\square$

Finally one should remark that it is very desirable to find a direct proof of these facts avoiding this cumbersome case-by-case analysis.

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